TRANSFORMATIONS AND SYMMETRY 1.2.1 – 1.2.5

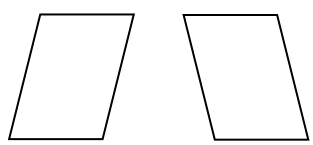
Studying transformations of geometric shapes builds a foundation for a key idea in geometry: congruence. In this introduction to transformations, the students explore three rigid motions: translations, reflections, and rotations. These explorations are done with tracing paper as well as with dynamic tools on the computer or other device. Students apply one or more of these motions to the original shape, creating its image in a new position without changing its size or shape. Rigid transformations also lead directly to studying symmetry in shapes. These ideas will help with describing and classifying geometric shapes later in the chapter.

See the Math Notes boxes in Lessons 1.2.2 and 1.2.4 for more information about rigid transformations.

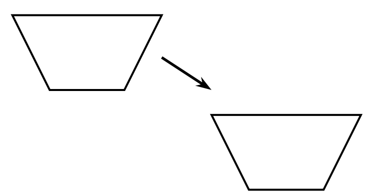
Example 1

Decide which rigid transformation was used on each pair of shapes below. Some may be a combination of transformations.

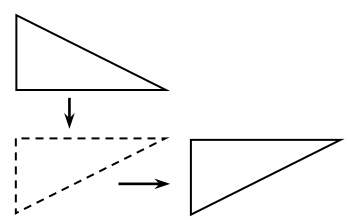
a.



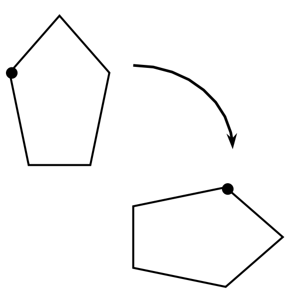
b.



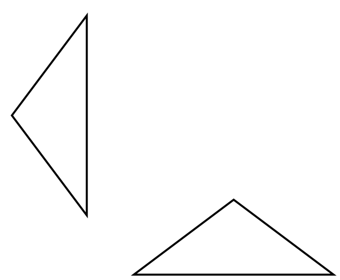
c.



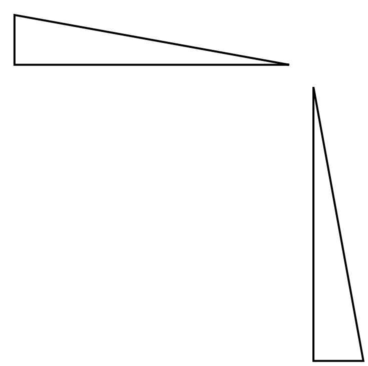
d.



e.



f.

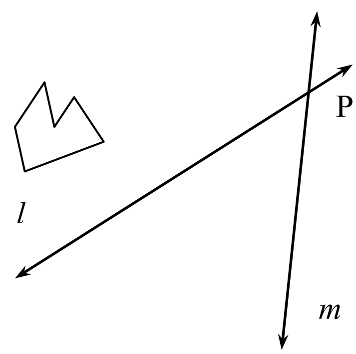


Identifying a single transformation is usually easy for students. In part (a), the parallelogram is reflected (flipped) across an invisible vertical line. (Imagine a mirror running vertically between the two figures. One figure would be the reflection of the other.) Reflecting a shape once changes its orientation. For example, in part (a), the two sides of the figure at left slant upwards to the right, whereas in its reflection at right, they slant upwards to the left. Likewise, the angles in the figure at left "switch positions" in the figure at right. In part (b), the shape is translated (or slid) to the right and down. The orientation remains the same, with all sides slanting the same.

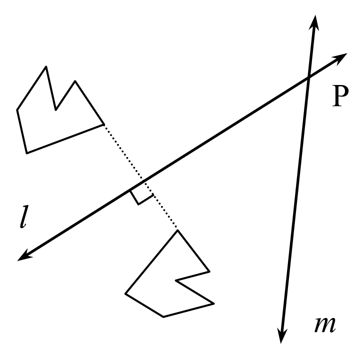
Part (c) shows a combination of transformations. First the triangle is reflected (flipped) across an invisible horizontal line. Then it is translated (slid) to the right. The pentagon in part (d) has been rotated (turned) clockwise to create the second figure. Imagine tracing the first figure on tracing paper, then holding the tracing paper with a pin at one point below the first pentagon, then turning the paper to the right 90°. The second pentagon would be the result. Some students might see this as a reflection across a diagonal line. The pentagon itself could be, but with the added dot (small circle), the entire shape cannot be a reflection. If it had been reflected, the dot would have to be on the corner below the one shown in the rotated figure. The triangles in part (e) are rotations of each other (90° again). Part (f) shows another combination. The triangle is rotated (the shortest side becomes horizontal instead of vertical) and reflected.

Example 2

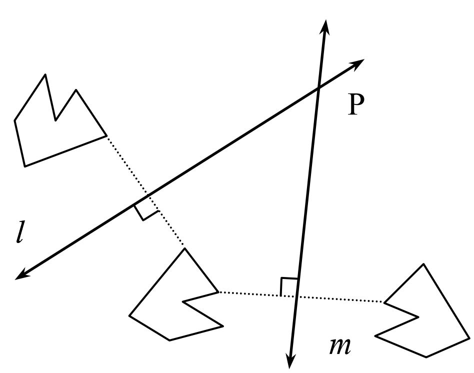
What will the figure at right look like if it is first reflected across line *l* and then the result is reflected across line *m*?



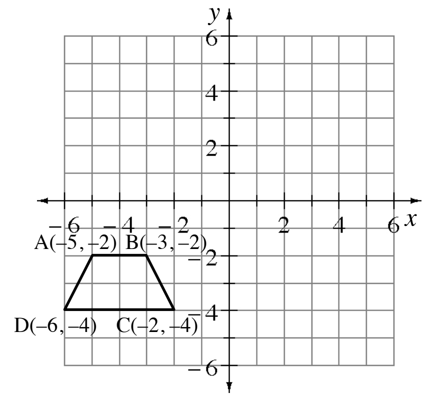
The first reflection is the new figure shown between the two lines. If we were to join each vertex (corner) of the original figure to its corresponding vertex on the second figure, those line segments would be perpendicular to line *l* and the vertices of (and all the other points in) the reflection would be the same distance away from *l* as they are in the original figure. One way to draw the reflection is to use tracing paper to trace the figure and the line *l*. Then turn the tracing paper over, so that line *l* is on top of itself. This will show the position of the reflection. Transfer the figure to your paper by tracing it. Repeat this process with line *m* to form the third figure by tracing.



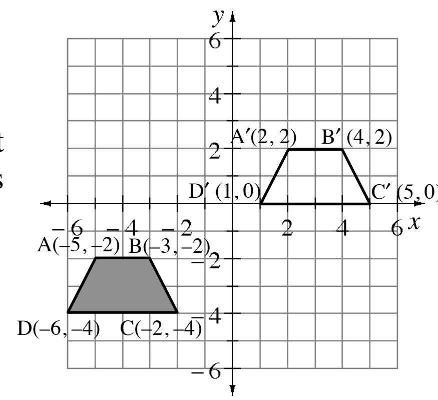
As students discovered in class, reflecting twice like this across two intersecting lines produces a **rotation** of the figure about the point P. Put the tracing paper back over the original figure to line *l*. Put a pin or the point of a pen or pencil on the tracing paper at point P (the intersection of the lines of reflection) and rotate the tracing paper until the original figure will fit perfectly on top of the last figure.



Example 3



The shape at right is trapezoid ABCD. Translate the trapezoid 7 units to the right and 4 units up. Label the new trapezoid A′B′C′D′ and give the coordinates of its vertices. Is it possible to translate the original trapezoid in such a way to create A″B″C″D″ so that it is a reflection of ABCD? If so, what would be the reflecting line? Will this always be possible for any figure?



Translating (or sliding) the trapezoid 7 units to the right and 4 units up gives a new trapezoid A′(2, 2), B′(4, 2), C′(5, 0), and D′(1, 0). If we go back to trapezoid ABCD, we now wonder if we can translate it in such a way that we can make it look as if it were a reflection rather than a translation. Since the trapezoid is symmetrical, it is possible to do so. We can slide the trapezoid horizontally left or right. In either case, the resulting figure would look like a reflection.

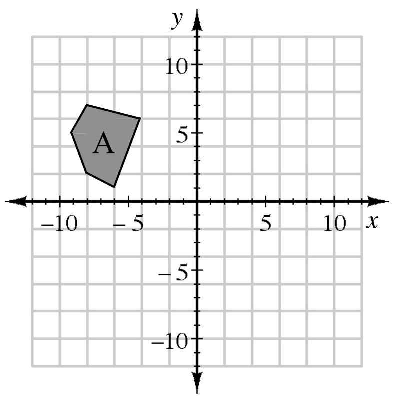
This will not always work. It works here because we started with an isosceles trapezoid, which has a line of symmetry itself. Students explored which polygons have lines of symmetry, and which have rotational symmetry as well. Again, they used tracing paper as well as technology to investigate these properties.

Exploring these transformations and symmetrical properties of shapes helps to improve students' visualization skills. These skills are often neglected or taken for granted, but much of mathematics requires students to visualize pictures, problems, or situations. That is why we ask students to "visualize" or "imagine" what something might look like as well as practice creating transformations of figures.

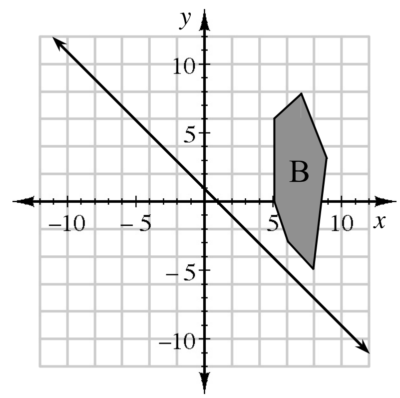
Problems

Perform the indicated transformation on each polygon below to create a new figure. You may want to use tracing paper to see how the figure moves.

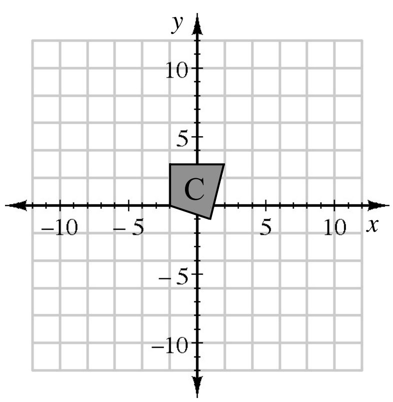
1. Rotate Figure A 90° clockwise about the origin.



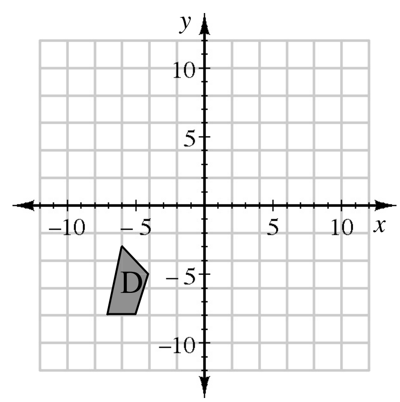
2. Reflect Figure B across line l.



3. Translate Figure C 6 units left.



4. Rotate Figure D 270° clockwise about the origin (0, 0).



For problems 5 through 20, refer to the figures below.

Figure A

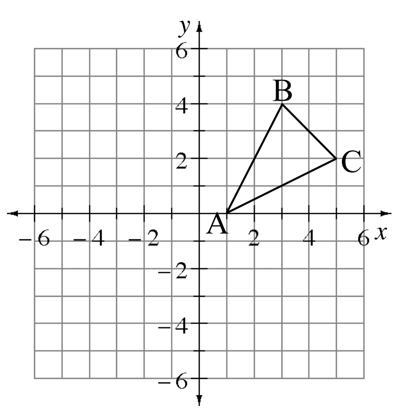


Figure B

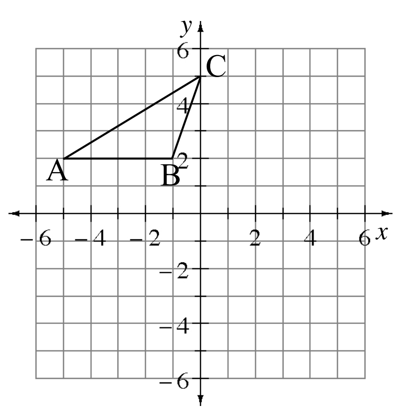
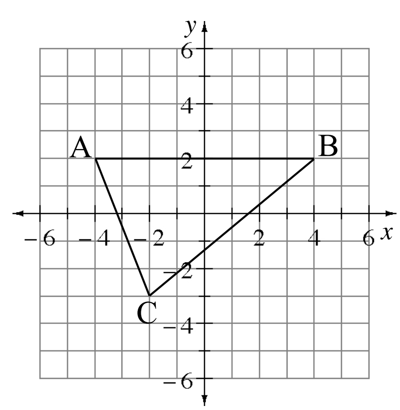


Figure C



State the new coordinates after each transformation.

5. Translate Figure A left 2 units and down 3 units.

6. Translate Figure B right 3 units and down 5 units.

7. Translate Figure C left 1 unit and up 2 units.

8. Reflect Figure A across the x-axis.

9. Reflect Figure B across the x-axis.

10. Reflect Figure C across the x-axis.

11. Reflect Figure A across the y-axis.

12. Reflect Figure B across the y-axis.

13. Reflect Figure C across the y-axis.

14. Rotate Figure A 90° counterclockwise about the origin.

15. Rotate Figure B 90° counterclockwise about the origin.

16. Rotate Figure C 90° counterclockwise about the origin.

17. Rotate Figure A 180° counterclockwise about the origin.

18. Rotate Figure C 180° counterclockwise about the origin.

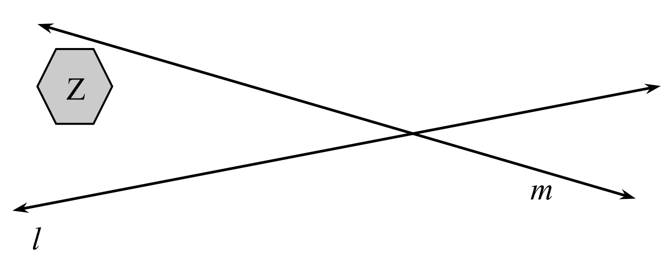
19. Rotate Figure B 270° counterclockwise about the origin.

20. Rotate Figure C 90° clockwise about the origin.

21. Plot the points A(3, 3), B(6, 1), and C(3, –4). Translate the triangle 8 units to the left and 1 unit up to create ΔA′B′C′. What are the coordinates of the new triangle?

22. How can you translate ΔABC in the last problem to put point A″ at (4, –5)?

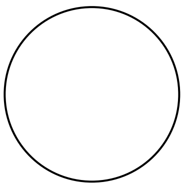
23. Reflect Figure Z across line l, and then reflect the new figure across line m. What are these two reflections equivalent to?

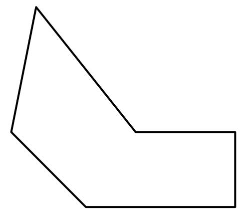


For each shape below, (*i*) draw all lines of symmetry, and (*ii*) describe its rotational symmetry if it exists.

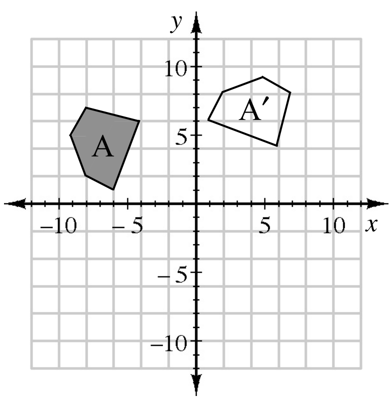
24. 

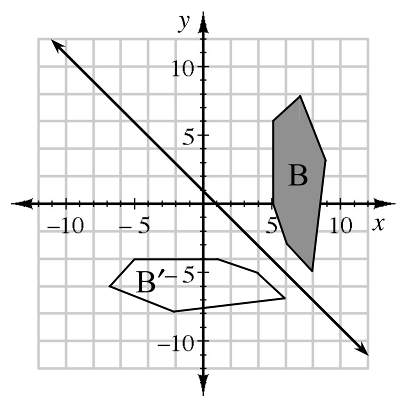
25. 

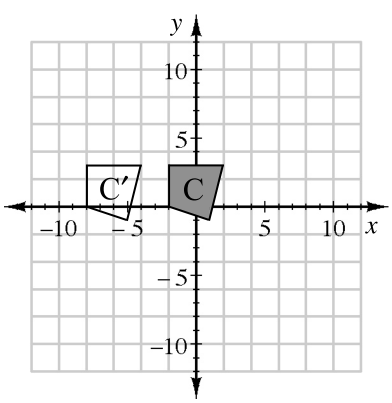
26. 

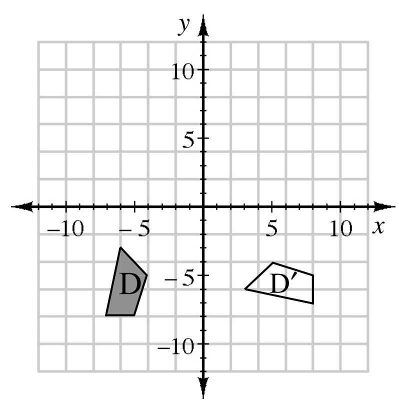
27. 

Answers

1. 

2. 

3. 

4. 

5. (–1, –3) (1, 1) (3, –1)

6. (–2, –3) (2, –3) (3, 0)

7. (–5, 4) (3, 4) (–3, –1)

8. (1, 0) (3, –4) (5, –2)

9. (–5, –2) (–1, –2) (0, –5)

10. (–4, –2) (4, –2) (–2, 3)

11. (–1, 0) (–3, 4) (–5, 2)

12. (5, 2) (1, 2) (0, 5)

13. (4, 2) (–4, 2) (2, –3)

14. (0, 1) (–4, 3) (–2, 5)

15. (–2, –5) (–5, 0) (–2, –1)

16. (–2, –4) (–2, 4) (3, –2)

17. (–1, 0) (–3, –4) (–5, –2)

18. (4, –2) (–4, –2) (2, 3)

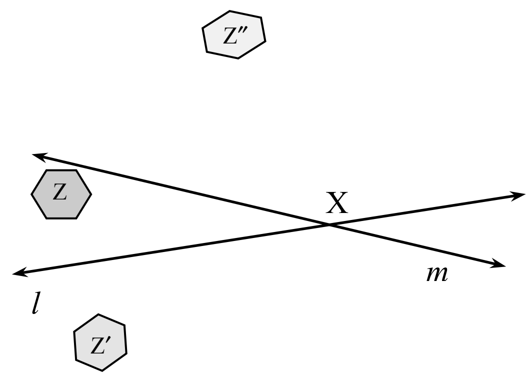
19. (2, 5) (2, 1) (5, 0)

20. (2, 4) (2, –4) (–3, 2)

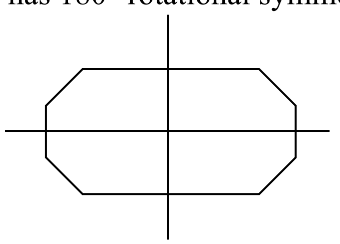
21. A′(5, –4) B′(–2, 2) C′(–5, –3)

22. Translate it 1 unit right and 8 units down.

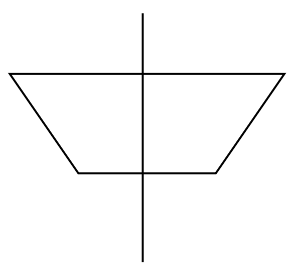
23. The two reflections are the same as rotating Z about point X.



24. This has 180° rotational symmetry.



25. The one line of symmetry. No rotational symmetry.



26. The circle has infinitely many lines of symmetry, every one of them illustrates reflection symmetry. It also has rotational symmetry for every possible degree measure.

27. This irregular shape has no lines of symmetry and does not have rotational symmetry, nor reflection symmetry.